A COMBINATORIAL HALF-INTEGRATION FROM WEIGHT SYSTEM TO VASSILIEV KNOT INVARIANT

SIMON WILLERTON*

Department of Mathematics and Statistics, University of Edinburgh,
James Clerk Maxwell Building, King's Buildings, Edinburgh, EH9 3JZ, Scotland.

Abstract

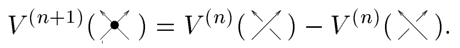
Generalizing a result of Domergue and Donato, it is shown that if the transition from degree n weight system to type n Vassiliev knot invariant is thought of as n steps of integration, then at least half of these steps can be performed canonically following symmetry considerations and a combinatorial formula is given for each of these steps.

Keywords: Vassiliev invariants, knots, combinatorics, weight systems.

1 Introduction

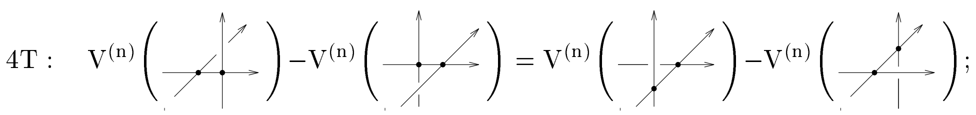
1.1 Preliminaries

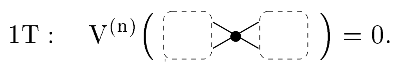
First recall the notions of Vassiliev invariants (see [1, 3] for more details). All invariants will take values in an abelian group G which has division by two (though it is not clear that this restriction is necessary). If X_n is the set of knots with n transversal self-intersections considered up to rigid vertex isotopy, then a knot invariant, V, (i.e. an invariant of X_0) can be extended to an invariant, $V^{(n)}$, of X_n by resolution of the double points, viz inductively:



Importantly, this is independent of the order in which the double points are resolved.

It can be shown easily that if V is a knot invariant then its extensions satisfy the following two relations:





*Current address: *Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3052, Australia.*

Definition. An invariant of X_n satisfying 4T and 1T will be called a *T-invariant*.

If P and R are invariants of X_n and X_{n+1} respectively and satisfy

$$R(\times\!\!\!\!\bullet\,) = P(\times) - P(\times)$$

then say that R is the derivative of P, that P is differentiable and that P is a first integral of R. If the derivative of P exists it will sometimes be denoted P'.

Remark. It is not saying here that all invariants have either derivative or first integral.

Definition. [9] If K and K' are two knots with n double points, then a *singular isotopy*, Φ, from K to K' is a sequence of isotopies of $\mathbf{R}^3$ and crossing changes which begin at K and end at K'. The *singular maps*, $\{\phi_i : i = 1, \ldots, t\}$, of Φ are the elements of X_{n+1} which correspond to the crossing changes. The *index*, $\epsilon(\phi_i)$, of a singular map ϕ_i is $+1$ or -1 respectively whether it corresponds to a positive to negative crossing change or *vice-versa*.

So if Φ is a singular isotopy from K to K' then it can be seen that for an invariant W with derivative V

$$W(K) - W(K') = \sum_{i=1}^{t} \epsilon(\phi_i) V(\phi_i).$$

A knot invariant, V, is said to be a Vassiliev invariant of type $n \in \mathbf{N}$ if $V^{(n+1)} = 0$, i.e. V vanishes on knots with greater than n double points. A T-invariant of X_n which differentiates to zero is called a degree n weight system. Thus a knot invariant V is of type n iff $V^{(n)}$ is a weight system.

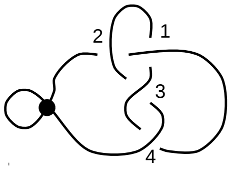
Bar-Natan and Kontsevich [1] showed that every weight system over $\mathbf{Q}$ or $\mathbf{C}$ can be integrated up to give a knot invariant. However their method was anything but calculable; this has been improved by the combinatorial methods (see e.g. [2]) stemming from Drinfeld's quasi-triangular quasi-Hopf algebras [5, 6]. One might prefer a more hands-on combinatorial approach for integration, utilizing the original discriminantal description of Vassiliev, rather than the seemingly unrelated quantum-algebraic methods.

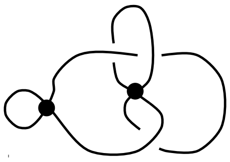
1.2 The Main Result

Domergue and Donato [4] proved that a degree n weight system could be integrated once by picking a regular projection of a knot with $(n-1)$ double points and evaluating the weight system on the knots obtained by smashing each of the crossings. This is generalized below.

A notion of regular projection holds for knots with double points as it does for non-singular knots. Let K be a knot with n double points and let π be a regular projection of K. Let $C = (a_1, \ldots, a_t)$ be an ordering of the crossings of π, with each a_i being either a positive or negative crossing, from which $\epsilon(a_i)$, the sign of the crossing, is respectively $+1$ or -1. Let (π, C, i) be the knot with $n+1$ double points obtained by switching the first $i-1$ crossings, making the ith crossing into a double point, and leaving the remaining crossings unchanged.

E.g.

$(\pi, C) =$ 

$(\pi, C, 3) =$  .

Remark. The point is that from a projection, π, of a knot K, with an ordering, C, of the crossings, a singular isotopy to the mirror image, $\overline{K}$ is obtained by flipping each of the crossings in order. So the singular maps of this singular isotopy are given precisely by the set $\{(\pi, C, i) : i = 1, \ldots, t\}$.

Definition. A *reflection-even* (or simply *even*) invariant, V, of X_n is one which is invariant under taking mirror images, i.e. $V(K) = V(\overline{K})$, and similarly a *reflection-odd* (or simply *odd*) invariant, W, is one which satisfies $W(K) = -W(\overline{K})$.

Remark. This notion of (reflection-)odd and even invariants should not be confused with Bar-Natan's notion of (inversion-)odd and even invariants which are concerned with switching the orientation of the knot rather than that of the ambient space.

The main result of this note is then:

Theorem. *(A) A differentiable, even T-invariant, V, of X_n has a unique odd first integral, W, and this is a T-invariant. Further, this is given on a regular projection, π, of K with ordering $C = (a_1, \ldots, a_t)$ by*

$$W(K) = \tfrac{1}{2} \sum\nolimits_{i=1}^{t} \epsilon(a_i) V(\pi, C, i).$$

(B) Any first integral of an odd T-invariant is even.

Two proofs are given of this. Section 3 consists of a proof based on that of [4] which uses the folk Reidemeister theorem for knots with double points. Section 4 presents a sketch of an alternative proof based on a theorem of Stanford (a full version of this is given in [11]).

Remark. It follows from the theorem that if the transition from degree n weight system to type n Vassiliev knot invariant is thought of as n steps of integration, then at least half of these steps can be performed canonically by the above method. This is because a weight system is an even T-invariant (as a singular isotopy can be constructed between K and $\overline{K}$) and so integrates to a unique odd invariant. This integrates, by whatever method, to an even T-invariant. The process continues inductively until a knot invariant is obtained. This explains the title of this note. (Note that this does not say that any weight system over G can be integrated up to a knot invariant.)

Thus the problem of integrating from a weight system to a knot invariant is reduced to the particular problem of integrating *odd* T-invariants.

2 Heuristic

If one integrates an invariant of X_n then there is indeterminacy, i.e. a constant of integration; in the case of integrating from a T-invariant to a T-invariant the constant of integration is a degree $n-1$ weight system.

Previously, when actuality tables were constructed these weight systems were fixed by picking some embedding respecting each chord diagram and fixing the value of W on these. However, there does not seem to be any canonical choice of embedding for each chord diagram.

If K is some knot with double points and some selected crossing then let K_+, K_- and K_X be K with the selected crossing respectively positive, negative, or smashed to a double point. If V is an even invariant and W a first integral of V, then thinking carefully about mirror images, the following holds:

$$\begin{aligned} W(K_+) - W(K_-) &= V(K_X) \;=\; V(\overline{K_X}) \;=\; V((\overline{K})_X) \\ &= W((\overline{K})_+) - W((\overline{K})_-) \\ &= W(\overline{K_-}) - W(\overline{K_+}). \end{aligned}$$

Whence

$$W(K_+) + W(\overline{K_+}) = W(K_-) + W(\overline{K_-}).$$

So defining $\hat{W}(K) = W(K) + W(\overline{K})$, it is seen that $\hat{W}$ is a T-invariant which is invariant under crossing changes, i.e. it is a weight system. This can then actually be identified as (twice) the constant of integration which may be canonically set to zero.

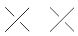
Perhaps there is a similar way to identify the constant of integration in the case of an odd invariant.

3 Proof of the Theorem

The proof given in this section utilizes the folk Reidemeister theorem for knots with double points: this is the notion, stated in [7] but not proved, that two knots with double points are equivalent up to rigid vertex isotopy if and only if they have regular projections which are equivalent via a set of five augmented Reidemeister moves. This combinatorial proof is modelled on the one in [4].

(A) The following are proved:
(i). W is independent of the ordering of crossings;
(ii). W is an invariant of X_{n-1};
(iii). W is a T-invariant;
(iv). W is a first integral of V;
(v). W is the unique *odd* first integral of V.

(i). Permutations of the ordering are generated by transpositions of adjacent crossings, so it suffices to prove that W is invariant under these. Consider a diagram with the following two crossings:

.

Let $C = (a_1, \ldots, a_t)$ be an ordering in which the left crossing is a_j and the right crossing is a_{j+1}; let $C' = (a'_1, \ldots, a'_t)$ be the ordering which is the same except that now the right crossing is a'_j and the left crossing is a'_{j+1}. As V is differentiable,

$$\begin{aligned} \epsilon(a_j)V(\quad\quad) - \epsilon(a_j)V(\quad\quad) &= \epsilon(a_j)\epsilon(a_{j+1})V'(\quad\quad) \\ &= \epsilon(a_{j+1})V(\quad\quad) - \epsilon(a_{j+1})V(\quad\quad), \end{aligned}$$

which gives

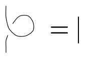
$$\epsilon(a_j)V(\pi, C, j) - \epsilon(a'_{j+1})V(\pi, C', j+1) = \epsilon(a'_j)V(\pi, C', j) - \epsilon(a_{j+1})V(\pi, C, j+1).$$

So

$$\begin{aligned} \sum \epsilon(a_i)V(\pi, C, i) &= \sum\nolimits_{i \neq j, j+1} \epsilon(a_i)V(\pi, C, i) \\ &\qquad + \epsilon(a_j)V(\pi, C, j) + \epsilon(a_{j+1})V(\pi, C, j+1) \\ &= \sum\nolimits_{i \neq j, j+1} \epsilon(a'_i)V(\pi, C', i) \\ &\qquad + \epsilon(a'_{j+1})V(\pi, C', j+1) + \epsilon(a'_j)V(\pi, C', j) \\ &= \sum \epsilon(a'_i)V(\pi, C', i). \end{aligned}$$

(ii). Here the augmented Reidemeister moves are required; the five are shown and the invariance of W under them is proved.

RI:

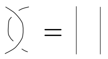


Let C be an ordering of the left hand diagram such that the selected crossing is a_1, then

$$\begin{aligned} W(\quad) &= \tfrac{1}{2}\epsilon(a_1)V(\quad) + \tfrac{1}{2}\sum_{1<i\leq t} \epsilon(a_i)V(\quad, C, i) \\ &= \tfrac{1}{2}\sum_{1<i\leq t} \epsilon(a_i)V(\,|\,, C, i) \\ &= W(\,|\,), \end{aligned}$$

as V satisfies 1T.

RII:

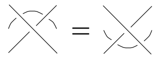


Let C be an ordering of the left hand diagram in which the top crossing is a_2 and the bottom one is a_1.

$$\begin{aligned} W(\quad) &= \tfrac{1}{2}\epsilon(a_1)V(\quad) + \tfrac{1}{2}\epsilon(a_2)V(\quad) + \tfrac{1}{2}\sum_{2<i\leq t} \epsilon(a_i)V(\quad, C, i) \\ &= \tfrac{1}{2}\sum_{2<i\leq t} \epsilon(a_i)V(\,|\quad|\,, C, i) \\ &= W(\,|\quad|\,), \end{aligned}$$

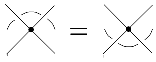
as the first two terms cancel and V satisfies RII.

RIII:



This is proved in a similar manner to the rest and left as an exercise.

RIV:

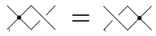


Let $C = (a_1, \ldots, a_t)$ be an ordering of the left hand diagram such that the top right crossing is a_1 and the top left crossing is a_2, and let $C' = (a'_1, \ldots, a'_t)$ be the ordering of the right hand diagram such that the bottom left crossing is a'_1, the bottom right crossing is a'_2, and all the other crossings are ordered as in C. Then $\epsilon(a_1) = \epsilon(a'_1)$, $\epsilon(a_2) = \epsilon(a'_2)$, and

$$\begin{aligned} W(\quad) &= \tfrac{1}{2}\epsilon(a_1)V(\quad) + \tfrac{1}{2}\epsilon(a_2)V(\quad) + \tfrac{1}{2}\sum_{2<i\le t} \epsilon(a_i)V(\quad, C, i) \\ &= \tfrac{1}{2}\epsilon(a'_1)V(\quad) + \tfrac{1}{2}\epsilon(a'_2)V(\quad) + \tfrac{1}{2}\sum_{2<i\le t} \epsilon(a_i)V(\quad, C', i) \\ &= W(\quad), \end{aligned}$$

as V satisfies 4T.

RV:



Let C be an ordering of the left hand diagram in which the crossing shown is a_1, and C' be the obvious corresponding ordering of the right hand diagram. Then $\epsilon(a_1) = \epsilon(a'_1)$, and

$$\begin{aligned} W(\quad) &= \tfrac{1}{2}\epsilon(a_1)V(\quad) + \tfrac{1}{2}\sum_{1<i\le t} \epsilon(a_i)V(\quad, C, i) \\ &= \tfrac{1}{2}\epsilon(a'_1)V(\quad) + \tfrac{1}{2}\sum_{1<i\le t} \epsilon(a'_i)V(\quad, C, i) \\ &= W(\quad). \end{aligned}$$

(iii). That W satisfies the 1T relation is immediate from the fact that V satisfies it. The satisfaction of 4T is proved similarly and again left as an exercise.

(iv). To show $W(\quad) - W(\quad) = V(\quad)$.

Let $C = (a_1, \ldots, a_t)$ be an ordering of the crossings where the selected crossing is last, and let C'' be the ordering obtained by just moving the selected crossing from the end to the beginning of the ordering. Then the following hold:

$$\begin{aligned} W(\quad) &= \tfrac{1}{2}\sum_{i<t} \epsilon(a_i)V(\quad, C, i) + \tfrac{1}{2}V(\quad), \\ W(\quad) &= -\tfrac{1}{2}V(\quad) + \tfrac{1}{2}\sum_{i>1} \epsilon(a''_t)V(\quad, C'', i) \\ &= -\tfrac{1}{2}V(\quad) + \tfrac{1}{2}\sum_{i>1} \epsilon(a_{t-1})V(\quad, C, i-1). \end{aligned}$$

So

$$W(\times) - W(\times) = \tfrac{1}{2}(V(\overline{\times\!\!\!\!\bullet}) + V(\times\!\!\!\!\bullet)) = V(\times\!\!\!\!\bullet).$$

(v). If K is a knot with n double points, then let Φ be any singular isotopy from K to its mirror image $\overline{K}$, let the singular points of Φ be $\{\phi_i : i = 1, \ldots, t\}$. Then $W(K) - W(\overline{K}) = \sum \epsilon(\phi_i)V(\phi_i)$, i.e. is determined by V, so if an odd first integral exists it is unique. The construction proves existence.

(B) If K is a knot with n double points and with the above notation, $\overline{\Phi}$ is a singular isotopy from $\overline{K}$ to K, the singular points are $\{\overline{\phi_i} : i = 1, \ldots, t\}$ and the indices are $\epsilon(\overline{\phi_i}) = -\epsilon(\phi_i)$.

If W is an odd invariant with first integral Y then

$$\begin{aligned} Y(K) - Y(\overline{K}) &= \sum_{i=1}^{t} \epsilon(\phi_i)W(\phi_i) = \sum_{i=1}^{t} -\epsilon(\phi_i)W(\overline{\phi_i}) = \sum_{i=1}^{t} \epsilon(\overline{\phi_i})W(\overline{\phi_i}) \\ &= Y(\overline{K}) - Y(K), \end{aligned}$$

whence $Y(K) = Y(\overline{K})$.

4 Alternative proof of Theorem

A theorem of Stanford [9, Lemma 2.2] (see [8, Theorem 6.1] for a clearer statement and a generalization) states that an invariant V of X_n is integrable if and only if it it is a differentiable T-invariant. This is purely an existence theorem and does not imply that the first integral is itself a T-invariant. This can be put to use in the symmetric case as follows.

Let V be an even differentiable T-invariant, then by Stanford's theorem there exists an invariant U which is a first integral of V. Define the invariant Y via

$$Y(K) = \tfrac{1}{2}(U(K) - U(\overline{K})),$$

This is clearly odd, and a little double point resolving together with the fact that V is even, gives that Y is also a first integral of V. By the uniqueness argument for odd first integrals given in the last section, it must be that Y is given by the formula in the statement of the theorem, i.e. $Y = W$, so W is a well defined invariant. The method of the last section can then be used to show that W satisfies 1T and 4T. (The details of this are also to be found in [11].)

Acknowledgements

Thanks go to the following: Raymond Lickorish for bringing [4] to the attention of the author; Dror Bar-Natan for comments about nomenclature; Steve Meeker for asking questions in the pub about mirror images; Joan Birman and the referee for pointing out the folk status of the Reidemeister theorem analogue for knots with double points; and Le Tu Quoc Thang and Matt Greenwood for comments about earlier versions of this note. The author was funded by an EPSRC studentship.

References

[1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) no. 2, 423–472.

[2] ....., *Non-Associative Tangles*, in *Geometric topology* (proceedings of the Georgia International topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[3] J. Birman and X. S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270.

[4] M. Domergue and P. Donato, *Integrating a weight system of order n to an invariant of $(n-1)$-singular knots*, J. Knot Theory Ramifications **5** (1996) 23–35.

[5] V. G. Drinfeld, *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990) no. 6, 1419–1457.

[6] ....., *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Leningrad Math. J. **2** (1991) no. 4, 829–860.

[7] L. H. Kauffman, *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. **311** (1989), 697–710.

[8] X.-S. Lin, *Finite type link invariants of 3-manifolds*, Topology **33** (1994) 25–71.

[9] T. Stanford, *Finite-type invariants of knots, links, and graphs*, Topology **35** (1996) 1027–1050.

[10] V. A. Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, Trans. of Math. Mono. **98**, Amer. Math. Soc., Providence, 1992.

[11] S. Willerton, *On the Vassiliev invariants for knots and for pure braids*, PhD. thesis, University of Edinburgh, 1997.