ON THE FIRST TWO VASSILIEV INVARIANTS

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ABSTRACT

The values that the first two Vassiliev invariants take on prime knots with up to fourteen crossings are considered. This leads to interesting fish-like graphs. Several results about the values taken on torus knots are proved.

RESUME

Les valeurs que prennent les deux premiers invariants de Vassiliev sur les noeuds premiers avec jusqu'à quatorze croisements sont consideré. Cela donne d'intéressants graphes en forme de poissons. Plusieurs resultats sur les valeurs prises sur les noeuds toriques sont demontré.

'First the fish must be caught.'
That is easy: a baby, I think, could have caught it.

— THE RED QUEEN, *Through the Looking Glass.*

INTRODUCTION.

The two simplest non-trivial Vassiliev knot invariants (see [18, 3]) are of type two and type three. These invariants have been studied from various angles: for instance, combinatorial formulæ for evaluating them have been derived, and simple bounds in terms of crossing number have been obtained (see e.g. [12, 9, 19]). In this work they are examined from the novel point of view of the actual values that they take on knots of small crossing number. For instance, one can ask how accurate the known bounds are, as in Section 1. When looking at this question I was led to plot the values of these invariants which revealed the interesting "fish" plots of Section 2: these pictures form the focus of this note. Various ideas which arise from these graphs can be answered for torus knots and this is done is Section 3. The final section presents some problems and further questions.

1. ON v_2 AND v_3.

The space of additive invariants of type three is two dimensional. By "the first two Vassiliev invariants" is meant the elements of a basis $\{v_2, v_3\}$ of this space. The invariants v_2 and v_3 can be defined canonically in the following fashion. The space of additive invariants of type three splits into the direct sum of type three invariants which do not distinguish mirror image knots and the type three invariants which differ by a factor of minus one on mirror image knots. Pick the vector in each of these one dimensional spaces which takes

Crossing number	3	4	5	6	7	8	9	10	11	12
Maximum $\lvert v_2\rvert$	1	1	3	2	6	5	10	9	15	14
Bound on $\lvert v_2\rvert$	1.5	2	5	7.5	11.5	14	18	22.5	27.5	33
Maximum $\lvert v_3\rvert$	1	0	5	1	14	10	30	25	55	49
Bound on $\lvert v_3\rvert$	1.5	6	15	30	57.5	84	126	180	247.5	330

TABLE 1. Comparing actual maxima and minima of $|v_2|$ and $|v_3|$ with the bounds of Section 1.

the value one on the positive trefoil. The one which is invariant under taking mirror images is of type two and will be denoted v_2; the other will be denoted v_3.

The invariant v_2 has appeared in various guises previously in knot theory: it is the coefficient of z^2 in the Conway polynomial and its reduction modulo two is the Arf invariant. Both v_2 and v_3 can be obtained from the Jones polynomial in the following fashion. If $J(q)$ is the Jones polynomial of a knot K and $J^{(n)}(q)$ denotes the n^{th} derivative with respective to q, then

$$v_2(K) = -\tfrac{1}{6}J^{(2)}(1), \qquad v_3(K) = -\tfrac{1}{36}\left(J^{(3)}(1) + 3J^{(2)}(1)\right).$$

Combinatorial formulæ for v_2 and v_3 can be given in terms of Gauß diagram formulæ—the reader is referred to [12, 19]. From the combinatorial formulæ, it is straight forward to obtain simple bounds for v_2 and v_3 in terms of the crossing number, c, of the knot, K: namely

$$|v_2(K)| \leq \tfrac{1}{4}c(c-1), \qquad |v_3(K)| \leq \tfrac{1}{4}c(c-1)(c-2).$$

The first of these bounds was obtained by Lin and Wang [10] and led Bar-Natan [2] to prove that any type n invariant is bounded by a degree n polynomial in the crossing number — this also follows from Stanford's algorithm [14] for calculating Vassiliev invariants. The bound for v_3 was obtained in [19] by utilizing Domergue and Donato's integration [6] of a type three weight system.

It is natural to ask how sharp these bounds are, and it is this question that motivated this work. Stanford has calculated Vassiliev invariants up to order six for the prime knots up to ten crossings, the programs and data files of which are available as [13]; Thistlethwaite has calculated various polynomials for knots up to fifteen crossings, these are available in the `knotscape` program [7] and also up to twelve crossings the data are available via [16]. Using these data, one can compare the bounds on $|v_2|$ and $|v_3|$ given above, with the actual maximum attained for each crossing number — this comparison is made in Table 1. It is seen that in this range of crossing numbers, the bounds are not particularly tight.

By looking at the raw data, one can see that for odd crossing number $(2b+1)$, the maxima are achieved precisely by the $(2, 2b+1)$-torus knot, and that this dominates the v_2 and v_3 of the $(2b+1)$-crossing knots as well. Letting $T(p,q)$ be the knot type of the (p,q)-torus knot, Alvarez and Labastida [1] (see also Section 3 below) give explicitly for crossing number $c = 2b+1$,

$$v_2\left(T\left(2,c\right)\right) = (c^2-1)/8, \qquad v_3\left(T\left(2,c\right)\right) = c(c^2-1)/24.$$

One could conjecture that these give bounds on v_2 and v_3.

2. Plots for knots with up to fourteen crossings.

Having stared at the raw data of Stanford for sufficiently long to start noticing patterns, I was led to plot v_2 against v_3 for knots of each crossing number up to crossing number fourteen. These plots are contained in Figure 1 and Figure 2. The symmetry in the v_2-axis is expected, as this is just the effect of taking the mirror image of the knots. The "fish" shape of these plots is not expected! This shape suggests some bound of the form

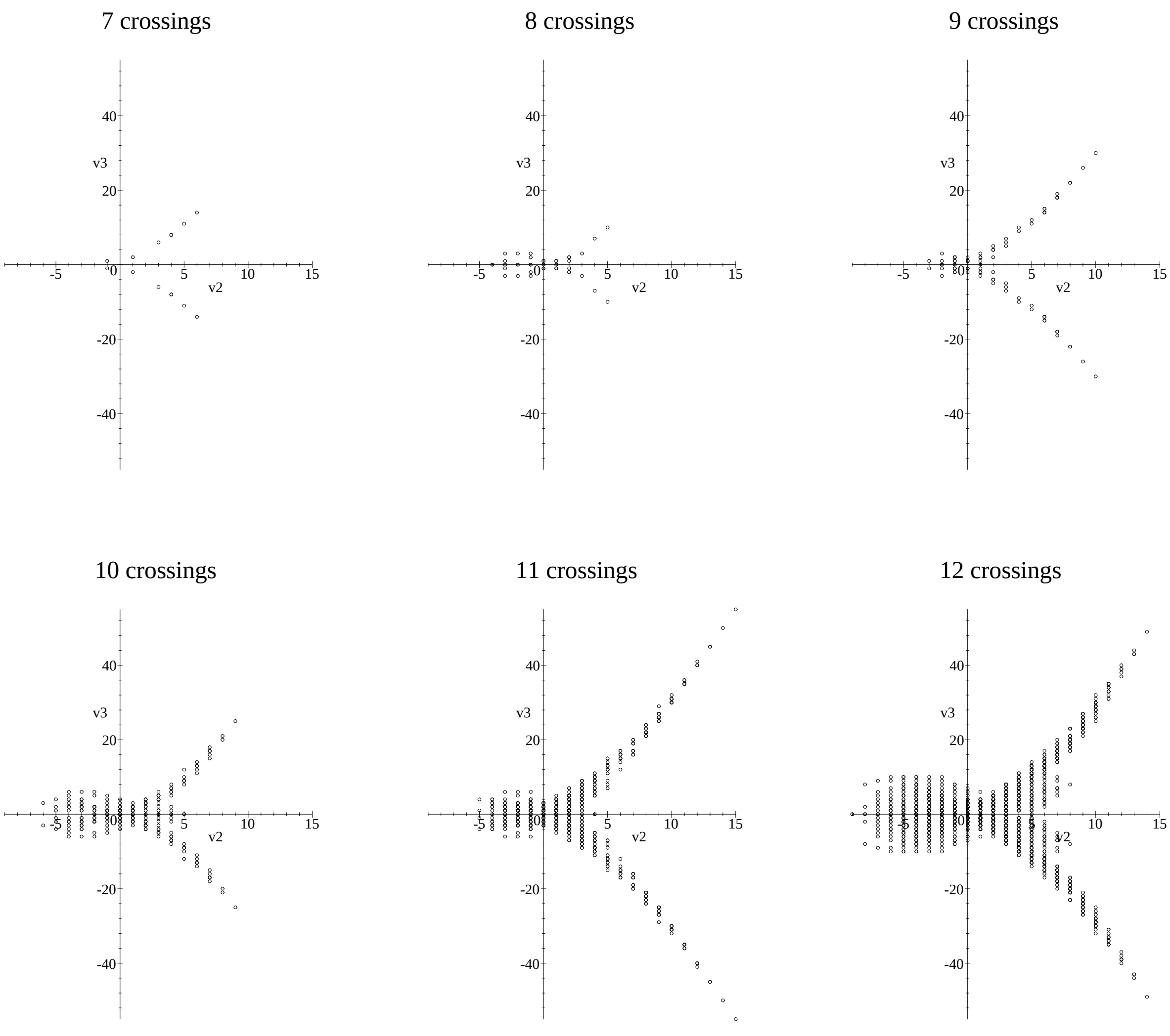

Figure 1. Plots by crossing number of v_2 and v_3 for the prime knots up to twelve crossings.

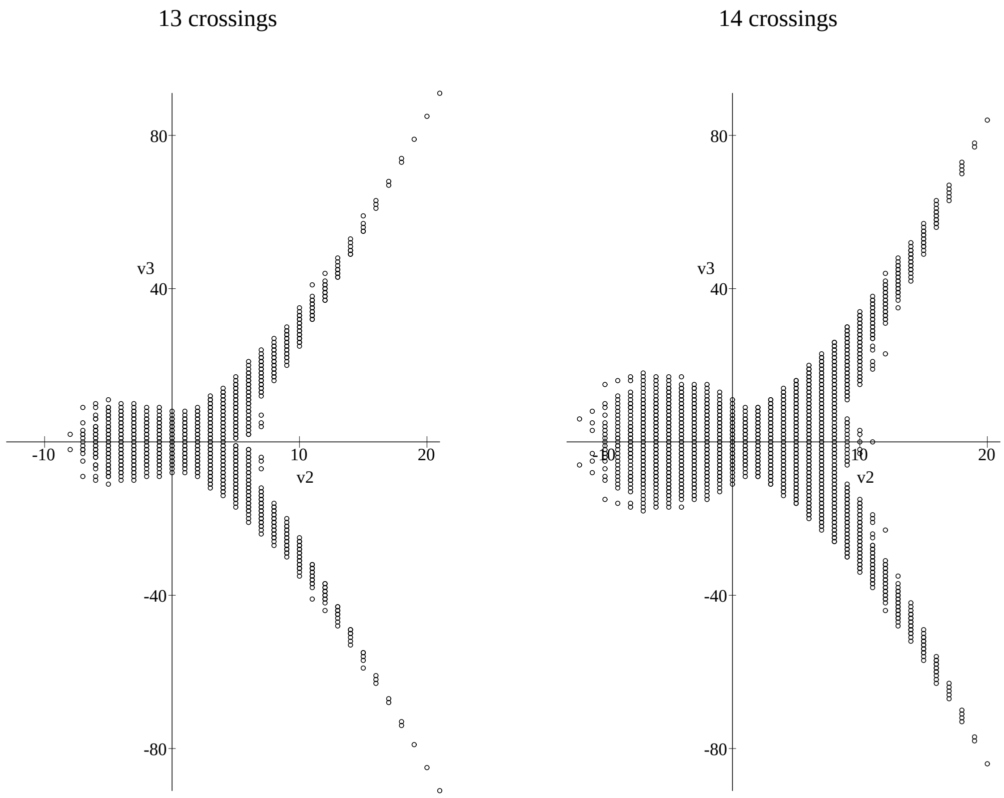

FIGURE 2. Plots by crossing number of v_2 and v_3 for the prime knots with thirteen and fourteen crossings.

$$\text{cubic in } v_2(K) \leq (v_3(K))^2 \leq \text{another cubic in } v_2(K).$$

Such bounds, independent of crossing number do in fact exist for torus knots, as will be seen below. However, this cannot be the case in general (unless the bounds depend on the crossing number): two reasons are as follows.

Firstly, consider the sequence of Whitehead doubles of the unknot, $\{Wh(i)\}_{i\in\mathbb{Z}}$ (see Figure 3). Table 2 gives the value of v_2 and v_3 on these for a range of i. It follows from the theorem of Dean [5] and Trapp [17] on twist sequences that a type n invariant evaluated on the Whitehead doubles is a polynomial in i of degree at most[1] n. A glance at Table 2 suffices to deduce that $v_2(Wh(i)) = i$ and $v_3(Wh(i)) = \frac{1}{2}i(i+1)$. Thus there is a sequence of knots (all except the unknot having unknotting number equal to one) which maps into the (v_2, v_3)-plane as a nice quadratic. This contradicts any bounds of the above form.

Secondly, for any $(a,b) \in \mathbb{Z}^2$ one can obtain a prime (alternating) knot with (v_2, v_3) equal to (a,b) in the following manner: connect sum suitably many positive and negative trefoil

[1]By an observation of Lin in this case it must be of degree at most $n-1$.

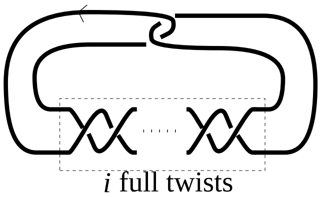

FIGURE 3. The ith twisted Whitehead double of the unknot, $Wh(i)$. For i negative, i full twists means $-i$ negative twists.

i	-3	-2	-1	0	1	2	3	4
$Wh(i)$	8_1	6_1	4_1	0_1	3_1	5_2	7_2	9_2
$v_2\left(Wh(i)\right)$	-3	-2	-1	0	1	2	3	4
$v_3\left(Wh(i)\right)$	3	1	0	0	1	3	6	10

TABLE 2. The values of v_2 and v_3 on the twisted Whitehead doubles of the unknot. The knot notation, e.g. 3_1, refers to Alexander-Briggs notation (see [4]).

knots (with $(v_2, v_3) = (1, \pm 1)$) and figure eight knots (with $(v_2, v_3) = (-1, 0)$), to obtain a composite knot with $(v_2, v_3) = (a, b)$, then Stanford [15] gives a method for constructing a prime knot with the same v_2 and v_3.

There does appear to be a qualitative difference between the pictures for odd and even crossing numbers in Figures 1 and 2. The even crossing number ones seem to be more concentrated in the 'body' of the 'fish' and the odd ones more in the 'tail'. Note that for each odd crossing number, c, there is the $(2, c)$-torus knot and the Whitehead double $Wh((c-1)/2)$ with a (v_2, v_3) of $((c-1)/2, (c^2-1)/8)$; and for even crossing number, c, there is the Whitehead double $Wh(1-c/2)$ with a (v_2, v_3) of $(1-c/2, (c-2)\,c/8)$. Also for up to twelve crossings (and it is conjectured for all knots), the amphicheiral knots — that is those equivalent to their mirror image, and hence with $v_3 = 0$ — all have even crossing number.

3. TORUS KNOTS.

The purpose of this section is to show that the torus knots map into the (v_2, v_3)-plane in a nice manner. In particular they satisfy cubic bounds of the form described above, implying that they lie on the tails of the fish; further, torus knots of the same unknotting number, or crossing number, lie on nice curves in the (v_2, v_3)-plane. The results of this section are summarized diagrammatically in Figure 4.

For p and q coprime let $T(p, q)$ be the knot type of the (p, q)-torus knot. Then $T(p, q)$ is the unknot if and only if p or q is ± 1, and for $T(p, q)$ nontrivial, $T(p, q)$ is the same knot as $T(p', q')$ if and only if (p', q') equals one the following (p, q), (q, p), $(-p, -q)$, or $(-q, -p)$. Further $T(p, -q)$ is the mirror image of $T(p, q)$. See [4].

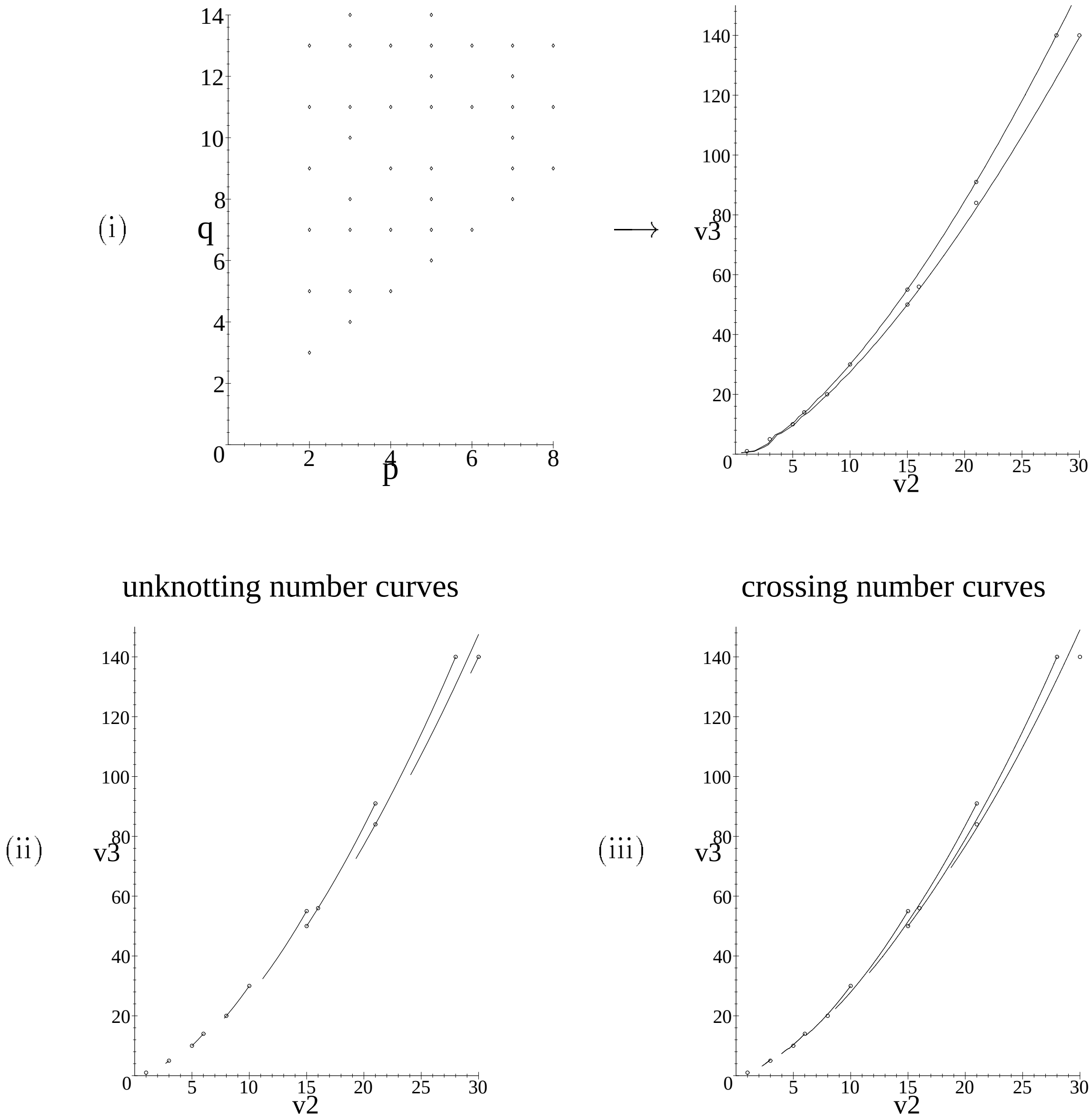

FIGURE 4. Torus knots in the (v_2, v_3)-plane: (i) mapping torus knots from the (p, q)-plane into the region of the (v_2, v_3)-plane given by Propositions 1 and 2; (ii) torus unknotting number curves for $u = 1, \ldots, 9$ (see Section 3.2); (iii) torus crossing number curves for $c = 3, 5, \ldots, 17$ (see Section 3.3).

The key to this section is the following pair of formulæ of Alvarez and Labastida [1].

$$v_2(T(p,q)) = \tfrac{1}{24}(p^2-1)(q^2-1), \qquad v_3(T(p,q)) = \tfrac{1}{144}pq(p^2-1)(q^2-1).$$

Note that these have the required properties under the symmetries of p and q mentioned above, and that these are integer valued on torus knots (i.e. when p and q are coprime). Also $T \mapsto (v_2(T), v_3(T))$ is injective for torus knots, that is to say torus knots are determined by their (v_2, v_3).

3.1. **Cubic bounds.** With the above formulæ of Alvarez and Labastida it is straightforward to prove bounds, for torus knots, of the form suggested in the last section.

PROPOSITION 1. *If T is a torus knot then*

$$\tfrac{2}{3}v_2(T)^3 + \tfrac{1}{3}v_2(T)^2 \le v_3(T)^2 \le \tfrac{8}{9}v_2(T)^3 + \tfrac{1}{9}v_2(T)^2.$$

Further, the right hand bound is tight in the sense that there exist torus knots with arbitrarily large v_2 and v_3 such that equality holds.

Proof. Suppose that T is a (p,q)-torus knot then

$$\begin{aligned} v_3(T)^2 - \tfrac{2}{3}v_2(T)^3 &= \left(\tfrac{1}{144}pq(p^2-1)(q^2-1)\right)^2 - \tfrac{2}{3}\left(\tfrac{1}{24}(p^2-1)(q^2-1)\right)^3 \\ &= \tfrac{1}{12^4}(p^2-1)^2(q^2-1)^2[p^2+q^2-1] \\ &\ge \tfrac{1}{12^3}(p^2-1)^2(q^2-1)^2 \qquad\qquad \text{as } p^2+q^2 \ge 13 \\ &= \tfrac{1}{3}v_2(T)^2, \end{aligned}$$

hence the first inequality (with equality only in the case of torus knots).

For the second,

$$\begin{aligned} \tfrac{8}{9}v_2(T)^3 - v_3(T)^2 &= \tfrac{8}{9}\left(\tfrac{1}{24}(p^2-1)(q^2-1)\right)^3 - \left(\tfrac{1}{144}pq(p^2-1)(q^2-1)\right)^2 \\ &= \tfrac{1}{4.27}\left[\tfrac{1}{24}(p^2-1)(q^2-1)\right]^2\left\{4(p^2-1)(q^2-1)-3p^2q^2\right\} \\ &= \tfrac{1}{4.27}v_2(T)^2\left\{(p^2-4)(q^2-4)-12\right\} \\ &\ge \tfrac{1}{4.27}v_2(T)^2\{-12\} = -\tfrac{1}{9}v_2(T)^2, \end{aligned}$$

and note that equality occurs precisely when T is a $(2,q)$-torus knot. $\square$

Although the left hand bound has the correct asymptotic behaviour, for a tight bound a different form of cubic is required.

PROPOSITION 2. *For a torus knot T,*

$$\tfrac{2}{3}v_2(T)^3 + \tfrac{1}{3}v_2(T)v_3(T) \le v_3(T)^2,$$

and this bound is tight in the sense of the previous proposition.

Proof. Using the notation of the previous proof,

$$\begin{aligned} v_3(T)^2 - \tfrac{2}{3}v_2(T)^3 - \tfrac{1}{3}v_2(T)v_3(T) &= \tfrac{1}{36.24^2}(p^2-1)^2(q^2-1)^2\left((p-q)^2-1\right) \\ &\ge 0, \end{aligned}$$

with equality if and only if T is a $(p,p+1)$ torus knot. $\square$

Given that half the torus knots (those with positive v_3) can be thought of as lying in the region $q > p > 0$ in the (p,q)-plane, these bounds are not surprising. Graphically this can be seen in Figure 4.

3.2. **Torus knots and unknotting number.** By Kronheimer and Mrowka's [8] positive solution to the Milnor conjecture the following formula is known for the unknotting number, u, of torus knots:

$$u(T(p,q)) = \tfrac{1}{2}\left(|p|-1\right)\left(|q|-1\right).$$

As a consequence, the following easily verifiable relationship is obtained:

Proposition 3. *For a torus knot T,*

$$v_2(T)^2 + \tfrac{1}{6}u(T)(u(T)-1)v_2(T) = u(T)|v_3(T)|,$$

and given $v_2(T)$ and $v_3(T)$ then $u(T)$ is the smaller of the two roots. □

So for a fixed unknotting number, the torus knots lie on a quadratic in the (v_2, v_3)-plane (c.f. the Whitehead knots in Section 2). This is pictured in Figure 4. The segments of curves shown were chosen by the following proposition.

Proposition 4. *For a torus knot T,*

$$\tfrac{1}{2}u(T)(u(T)+1) \geq v_2(T) \geq \tfrac{1}{6}u(T)\left(u(T) + \sqrt{8u(T)+1} + 2\right),$$

and both bounds are tight.

Proof. If T is a (p,q)-torus knot, then a minimal amount of manipulation gives

$$\begin{aligned}\tfrac{1}{2}u(T)(u(T)+1) - v_2(T) &= \tfrac{1}{12}(|p|-1)(|q|-1)(|p|-2)(|q|-2)\\ &\geq 0,\end{aligned}$$

with equality if and only if T is a $(2,q)$-torus knot.

For the right hand bound, firstly, let a and b be distinct positive integers, then $(a-b)^2 \geq 1$, so $(a+b)^2 \geq 4ab+1$ and thus $a+b \geq \sqrt{4ab+1}$, with equality precisely when a and b differ by one.

Now for T a (p,q)-torus knot,

$$\begin{aligned}&v_2(T) - \tfrac{1}{6}u(T)\left(u(T) + \sqrt{8u(T)+1} + 2\right)\\ &\qquad = \tfrac{1}{12}(|p|-1)(|q|-1)\left\{|p| + |q| - 2 - \sqrt{4(|p|-1)(|q|-1)+1}\right\}\\ &\qquad \geq 0,\end{aligned}$$

by putting $a = |p|-1,\ b = |q|-1$ in the above paragraph. Note that equality occurs precisely when T is a $(p, p+1)$-torus knot. □

Weakening the right hand bound to $v_2 \geq \frac{1}{6}u(T)\left(u(T)+5\right)$ and inverting the inequalities reveals the following corollary.

COROLLARY 5. *For a torus knot T,*

$$\sqrt{1+8v_2(T)}-1 \leq 2u(T) \leq \sqrt{24v_2(T)+25}-5,$$

and the left hand bound is tight (in the sense of Proposition 1). $\square$

3.3. **Torus knots and crossing number.** By the work of Murasugi [11], a similar formula is known for the crossing number, c, of torus knots:

$$c(T(p,q)) = |q|(|p|-1), \qquad \text{when } |p|<|q|.$$

This leads to the following relation;

PROPOSITION 6. *If T is a torus knot, and $\rho(T)=\left|\frac{6v_3(T)}{v_2(T)}\right|$ then*

$$24v_2(T)(c(T)-\rho(T))^2 = c(T)\left((c(T)-\rho(T))^2-1\right)\left(2\rho(T)-c(T)\right),$$

and

$$c(T)=\rho(T)-\tfrac{1}{2}\left(\sqrt{(\rho(T)-1)^2-24v_2(T)}+\sqrt{(\rho(T)+1)^2-24v_2(T)}\right).$$

Proof. This is easily verified; note that if T is a (p,q)-torus knot then $\rho(T)=|pq|$ and $c(T)-\rho(T)=|q|$. $\square$

This isn't as nice a relationship as with the unknotting number: for a fixed crossing number the relationship is a not particularly nice quartic between v_2 and v_3. However, the crossing number curves can still be graphed, as in Figure 4 — the length of arc segments plotted there being determined by the following proposition.

PROPOSITION 7. *For a torus knot T,*

$$\tfrac{1}{8}\left(c(T)^2-1\right) \geq v_2(T) \geq \tfrac{1}{24}c(T)\left(c(T)+1+2\sqrt{c(T)+1}\right),$$

and these bounds are tight (in the sense of Proposition 1).

Proof. Suppose that T is a (p,q)-torus knot with $q>p>0$ — this just avoids excessive modulus signs in the calculation — then for the left hand bound,

$$\begin{aligned}
\tfrac{1}{8}\left(c(T)^2-1\right)-v_2(T) &= \tfrac{1}{24}\left\{3\left([q(p-1)]^2-1\right)-(p^2-1)(q^2-1)\right\}\\
&= \tfrac{1}{24}\left\{2q^2p^2-6q^2p+4q^2+p^2-4\right\}\\
&= \tfrac{1}{24}(p-2)\left\{(2q^2+1)(p-1)+3\right\}\\
&\geq 0,
\end{aligned}$$

and equality occurs precisely when T is a $(2,q)$-torus knot.

For the right hand bound,

$$\begin{aligned}
&24v_2(T)-c(T)\left(c(T)+1+2\sqrt{c(T)+1}\right)\\
&\qquad = (p^2-1)(q^2-1)-q(p-1)\left(q(p-1)+1+2\sqrt{q(p-1)+1}\right)\\
&\qquad = (p-1)\left\{2q^2-q-1-p-2q\sqrt{qp-q+1}\right\},
\end{aligned}$$

and claim that this is non-negative and is zero precisely when $q = p + 1$.

To prove the claim, note

$$(q-1)^2 = q(q-1) - q - 1 \geq qp - q - 1$$

as $q - p - 1 \geq 0$, and so also

$$(q-1)^2 + \tfrac{2(q-1)(q-p-1)}{2q} + \left[\tfrac{q-p-1}{2q}\right]^2 \geq qp - q - 1 > 0,$$

thus, by taking square roots,

$$(q-1) + \tfrac{q-p-1}{2q} \geq \sqrt{qp - q - 1},$$

from which the claim follows on multiplying through by $2q$. □

Weakening the right hand bound to $v_2 \geq \frac{1}{24}c(c+5)$ and inverting, gives

Corollary 8. *For a torus knot T*

$$\tfrac{1}{24}\left(\sqrt{25 + 96 v_2(T)} - 5\right) \geq c(T) \geq 2\sqrt{8 v_2(T) + 1},$$

and the right hand bound is tight in the previous sense. □

4. Problems and further questions.

Problem 1. The invariants v_2 and v_3 form, in some sense, a canonical basis for the space of invariants of degree three. Can one find similarly, for example, a canonical basis for the space of type four, even, additive invariants? Can canonical bases be found for higher order invariants? The spaces of weight systems/chord diagrams split as direct sums of eigenspaces of cabling operations; can this be done similarly for the invariants?

Problem 2. Does the fish pattern persist in the graphs of knots with higher crossing number?

Problem 3. Is there some qualitative distinction between knots with odd and even crossing number which explains the perceived difference in the fish?

Problem 4. Is there any relationship with unknotting number, e.g. are there knots of arbitrary unknotting number with $(v_2, v_3) = (0, 0)$?

Problem 5. Can the cubic bounds between v_2 and v_3 for torus knots be extended to positive knots and negative knots? It is easy, for example, to get the linear bound $2v_3 \geq v_2 \geq 1$ for non-trivial positive knots using the Gauß diagram formulæ.

Problem 6. For a knot K with $(6|v_3(K)| - |v_2(K)|)^2 \geq 24 v_2(K)^3$, let $\rho(K) = 6|v_3(k)/v_2(K)|$ and then define the *pseudo-unknotting number*, $\tilde{u}(K)$, and the *pseudo-uncrossing number*, $\tilde{c}(K)$, by

$$\tilde{u}(K) := \tfrac{1}{2}\left(1 + \rho(K) - \sqrt{(1+\rho(K))^2 - 24 v_2(K)}\right);$$

$$\tilde{c}(K) := \rho - \tfrac{1}{2}\left(\sqrt{(1+\rho(K))^2 - 24 v_2(K)} + \sqrt{(1-\rho(K))^2 - 24 v_2(K)}\right).$$

For torus knots, the pseudo-unknotting and pseudo-crossing numbers coincide with the usual unknotting and crossing numbers. Do they have any meaning for other knots? Does the necessary bound for K have any topological interpretation?

As an example, consider the Whitehead knots $Wh(i)$, for $i > 0$ these all have unknotting number equal to one; in this case $\tilde{u}(Wh(1)) = 1$, and $\tilde{u}(Wh(i)) \to 2$ as $i \to \infty$.

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