Optimal transport and enriched categories

Simon Willerton
University of Sheffield

Cambridge
November 2021

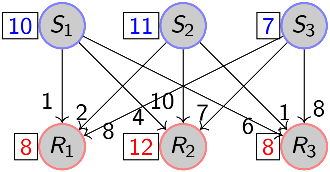
Optimal transport: primal problem

Suppliers $S_1, \ldots, S_s$, supply $\sigma_1, \ldots, \sigma_s$; receivers $R_1, \ldots R_r$, demand $\rho_1, \ldots \rho_r$. Cost of moving one unit from S_i to R_j is $k_{ij} \in \mathbb{R}_{\geq 0}$

$\mathbf{k_{ij}}$	R_1	R_2	R_3
S_1	1	4	6
S_2	2	10	1
S_3	8	7	8

	σ
S_1	10
S_2	11
S_3	7

	ρ
R_1	8
R_2	12
R_3	8



Definition (Primal optimal transport problem)

Given k, σ and ρ as above, find a transport plan $\{\alpha_{ij}\}_{ij}$ which minimizes

$$\text{cost} = \sum_{ij} k_{ij} \alpha_{ij}$$

subject to the supply and demand constraints:

$$\sum_i \alpha_{ij} \geq \rho_j; \quad \sum_j \alpha_{ij} \leq \sigma_i.$$

Linear programming duality

Definition (Primal linear programming problem)

Given $\{b_e\}$, $\{c_f\}$ and $\{A_{ef}\}$ find non-negative $x_1, \ldots, x_n$ which minimizes

$$\textstyle\sum_f c_f x_f$$

such that $\sum_f A_{ef} x_f \leq b_e$.

Definition (Dual linear programming problem)

Given $\{b_e\}$, $\{c_f\}$ and $\{A_{ef}\}$ find non-negative $y_1, \ldots, y_m$ which maximizes

$$\textstyle\sum_e y_e b_e$$

such that $\sum_e y_e A_{ef} \leq c_f$.

Theorem (Strong linear programming duality)

$$\inf_{\text{feasible } x} \textstyle\sum_f c_f x_f = \sup_{\text{feasible } y} \textstyle\sum_e y_e b_e.$$

Optimal transport: dual problem

Definition (Dual optimal transport problem)

Given k, σ and ρ, find prices $v_1, \dots, v_s$ and $u_1, \dots, u_r$ which maximize

$$\text{revenue} = \sum_i u_j\rho_j - \sum_i v_i\sigma_i$$

subject to the competitive pricing constraint: $u_j - v_i \leq k_{ij}$.

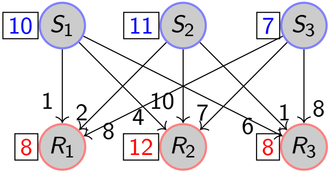
The Fable

A transportation company offers alternative transportation for the goods.
They have an unusual pricing structure.
They will buy the goods for unit price v_i *from supplier* S_i*.*
They will sell the goods for unit price u_j *to receiver* R_j*.*
The constraint ensures that they are cheaper than the original transportation.

Theorem

Minimum cost of primal problem equals maximum revenue of dual problem.

Example optimal solutions



	R_1	R_2	R_3	
S_1	5	5	0	10
S_2	3	0	8	11
S_3	0	7	0	7
	8	12	8	

$\text{cost} = 88$

	v
S_1	3
S_2	2
S_3	0

	u
R_1	4
R_2	7
R_3	3

$\text{revenue} = 88$

Duality within the dual

Suppose prices $\{v_i\}$ are chosen by the transportation company at suppliers. What are the highest feasible prices $\{\hat{v}_j\}$ to sell to the receivers?

$$\hat{v}_j := \min_i\{k_{ij} + v_i\}$$

Similarly for $\{u_j\}$ prices at receivers, the lowest feasible prices at supplier i is

$$\tilde{u}_i := \max(\max_j\{u_j - k_{ij}\}, 0).$$

This process is idempotent: $\hat{\tilde{\hat{v}}} = \hat{v}$.
If (v, u) is an optimal pricing plan then we can assume tightness, i.e. that

$$v = \tilde{u} \quad \text{and} \quad \hat{v} = u,$$

so we can look for optimal pricing plans in the centre of an 'adjunction':

$$\begin{tikzcd}\{\text{prices at suppliers, } v\} \arrow[r, shift left, "\widehat{\ }"] & \{\text{prices at recievers, } u\}. \arrow[l, shift left, "\widetilde{\ }"]\end{tikzcd}$$

Metric spaces and enriched categories

small category	$\mathcal{V}$-category	metric space
set of objects	set of objects	set of points
morphism set $\mathcal{C}(x,y)$	hom object $\mathcal{C}(x,y) \in \mathrm{Ob}(\mathcal{V})$	distance $\mathsf{d}(x,y) \in [0,\infty]$
$\mathcal{C}(x,y) \times \mathcal{C}(y,z) \to \mathcal{C}(x,z)$	$\mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \to \mathcal{C}(x,z)$	$\mathsf{d}(x,y) + \mathsf{d}(y,z) \geq \mathsf{d}(x,z)$
$\mathrm{id} \in \mathcal{C}(x,x)$	$\mathbb{1} \to \mathcal{C}(x,x)$	$0 = \mathsf{d}(x,x)$

We can define the notion of $\mathcal{V}$-category for any monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$. Eg

$$(\mathrm{Set}, \times, \{*\}), \quad (\mathrm{Truth}, \wedge, \mathrm{T}), \quad (\overline{\mathbb{R}}_+ = ([0,\infty], \geq), +, 0)$$

The notion of $\mathcal{V}$-functor for metric spaces is ‘distance non-increasing map’. Moreover, $\overline{\mathbb{R}}_+$ is complete, cocomplete, closed symmetric monoidal:

$$[x, y] = y \dot{-} x := \max(y - x, 0)$$

This allows, eg, definition of functor $\mathcal{V}$-categories.

$\mathcal{V}$-profunctors (for $\mathcal{V}$ sufficiently nice)

Profunctor $f\colon \mathcal{C} \nrightarrow \mathcal{D}$ for $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ means $f\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$.
Composition $\mathcal{C} \overset{f}{\nrightarrow} \mathcal{D} \overset{g}{\nrightarrow} \mathcal{E}$ defined by

$$g \circ f(c, e) := \int^{d} g(d, e) \otimes f(c, d) = \min_d (g(d, e) + f(c, d))$$

$\mathcal{V}$-profunctors (for $\mathcal{V}$ sufficiently nice)

Profunctor $f\colon \mathcal{C} \nrightarrow \mathcal{D}$ for $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ means $f\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$.
Composition $\mathcal{C} \overset{f}{\nrightarrow} \mathcal{D} \overset{g}{\nrightarrow} \mathcal{E}$ defined by

$$g \circ f(c,e) := \int^{d} g(d,e) \otimes f(c,d) = \min_d (g(d,e) + f(c,d))$$

These form a bicategory Prof which is ‘composition closed’ (aka biclosed), i.e., composition with fixed profunctor has a right adjoint.

$$\begin{tikzcd}
\mathrm{Prof}(\mathcal{C},\mathcal{D}) \arrow[r, bend left, "g\circ -"] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\mathcal{C},\mathcal{E}) \arrow[l, bend left, "g\triangleright -"] \arrow[r, bend right, "-\triangleleft f"'] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\mathcal{D},\mathcal{E}) \arrow[l, bend right, "-\circ f"']
\end{tikzcd}$$

$\mathcal{V}$-profunctors (for $\mathcal{V}$ sufficiently nice)

Profunctor $f\colon \mathcal{C} \nrightarrow \mathcal{D}$ for $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ means $f\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$.
Composition $\mathcal{C} \overset{f}{\nrightarrow} \mathcal{D} \overset{g}{\nrightarrow} \mathcal{E}$ defined by

$$g \circ f(c, e) := \int^{d} g(d, e) \otimes f(c, d) = \min_d(g(d, e) + f(c, d))$$

These form a bicategory Prof which is 'composition closed' (aka biclosed), i.e., composition with fixed profunctor has a right adjoint.

$$\begin{tikzcd}
\mathrm{Prof}(\mathcal{C}, \mathcal{D}) \arrow[r, bend left, "g\circ -"] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\mathcal{C}, \mathcal{E}) \arrow[l, bend left, "g\rhd -"] \arrow[r, bend right, "-\lhd f"'] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\mathcal{D}, \mathcal{E}) \arrow[l, bend right, "-\circ f"']
\end{tikzcd}$$

$$g \rhd h(c, d) := \int_{e} [g(d, e), h(c, e)] = \max_d(g(d, e) \mathbin{\dot{-}} h(c, e))$$

$\mathcal{V}$-profunctors (for $\mathcal{V}$ sufficiently nice)

Profunctor $f\colon \mathcal{C} \nrightarrow \mathcal{D}$ for $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ means $f\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$.
Composition $\mathcal{C} \overset{f}{\nrightarrow} \mathcal{D} \overset{g}{\nrightarrow} \mathcal{E}$ defined by

$$g \circ f(c, e) := \int^{d} g(d, e) \otimes f(c, d) = \min_d (g(d, e) + f(c, d))$$

These form a bicategory Prof which is 'composition closed' (aka biclosed), i.e., composition with fixed profunctor has a right adjoint.

$$\begin{tikzcd}
\mathrm{Prof}(\mathcal{C},\mathcal{D}) \arrow[r, bend left, "g\circ -"] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\mathcal{C},\mathcal{E}) \arrow[l, bend left, "g\triangleright -"] \arrow[r, bend right, "-\triangleleft f"'] \arrow[r, phantom, "\perp"] & \mathrm{Prof}(\mathcal{D},\mathcal{E}) \arrow[l, bend right, "-\circ f"']
\end{tikzcd}$$

$$g \triangleright h(c, d) := \int_{e} [g(d, e), h(c, e)] = \max_d (g(d, e) \mathbin{\dot{-}} h(c, e))$$

Note that if $*$ is the unit $\mathcal{V}$-category then

$$\mathrm{Prof}(*, \mathcal{D}) = \mathrm{Fun}(\mathcal{D}, \mathcal{V}) = \mathcal{V}^{\mathcal{D}}.$$

Modelling optimal transport with enriched categories

We enrich over ‘cost’ which we take to be non-negative real numbers.
Our $\mathcal{V}$-categories are generalized metric spaces.

Suppliers:	$\mathcal{S}$	discrete $\overline{\mathbb{R}}_+$-category
Receivers:	$\mathcal{R}$	discrete $\overline{\mathbb{R}}_+$-category
Transport cost:	$k\colon \mathcal{S} \nrightarrow \mathcal{R}$	$\overline{\mathbb{R}}_+$-profunctor
Prices at suppliers:	$v\colon \mathcal{S} \to \overline{\mathbb{R}}_+$	$\overline{\mathbb{R}}_+$-functor
Prices at receivers:	$u\colon \mathcal{R} \to \overline{\mathbb{R}}_+$	$\overline{\mathbb{R}}_+$-functor

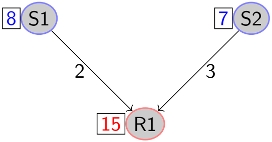
The price duality in the dual problem arises as the ‘Kan-type’ adjunction.

$$\begin{tikzcd} v \in \overline{\mathbb{R}}_+^{\mathcal{S}} \arrow[r, bend left, "k \circ -"] \arrow[r, phantom, "\perp"] & \overline{\mathbb{R}}_+^{\mathcal{R}} \ni u \arrow[l, bend left, "k \triangleright -"] \end{tikzcd}$$

This is categorically inevitable.
The tight price plans are those in the centre Z of this adjunction:

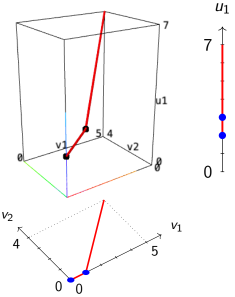
$$Z := \{(v, u) \in \overline{\mathbb{R}}_+^{\mathcal{S}} \times \overline{\mathbb{R}}_+^{\mathcal{R}} \mid k \circ v = u;\ v = k \triangleright u\}.$$

A first example

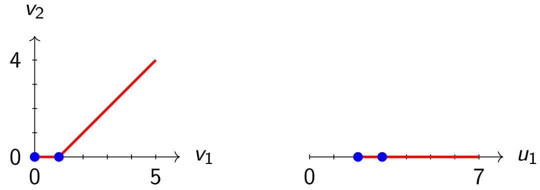


The profunctor is $k = \left(\begin{smallmatrix}2\\3\end{smallmatrix}\right)$.
Calculate the tight price plans:

$$Z \subset \overline{\mathbb{R}}_+^2 \times \overline{\mathbb{R}}_+^1$$

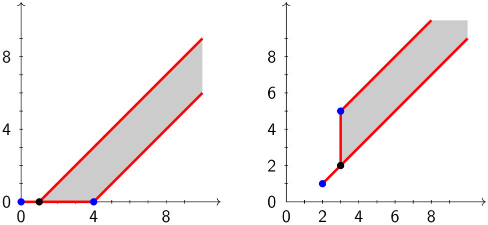


Here are the projections drawn in a more standard way.

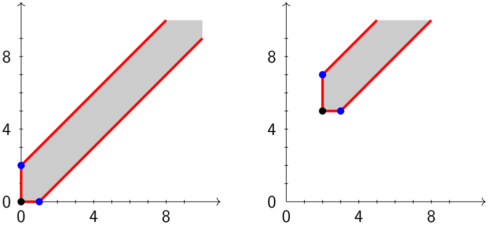


More examples

$\begin{pmatrix}2 & 1\\ 3 & 5\end{pmatrix}$

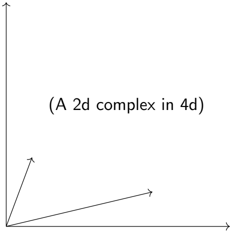


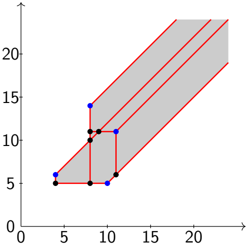
$\begin{pmatrix}2 & 7\\ 3 & 5\end{pmatrix}$



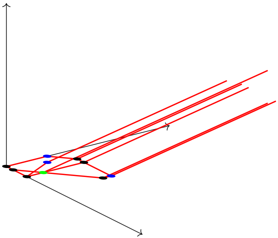
And more examples

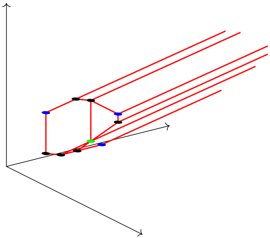
$$\begin{pmatrix} 4 & 6 \\ 10 & 5 \\ 11 & 11 \\ 8 & 14 \end{pmatrix}$$





$$\begin{pmatrix} 1 & 4 & 6 \\ 2 & 10 & 1 \\ 8 & 7 & 8 \end{pmatrix}$$





Properties of quantale-enriched adjunctions

Every adjunction is idempotent

$$RLR = R \quad \text{and} \quad LRL = L.$$

Can use standard adjunction properties, e.g. left adjoints preserve colimits.

$\mathcal{V}^{\mathcal{C}}$: colimits are pointwise product and copowers.
In our case this means that $\mathrm{Fix}(RL)$ is the tropical $(\min, +)$ span of the rows of k.

$$\begin{tikzcd}
& \mathcal{C} \times \mathcal{D} & \\
& Z(L \dashv R) \arrow[u, phantom, "\cup"] \arrow[dl, "\pi_{\mathcal{C}}", "\sim"'] \arrow[dr, "\pi_{\mathcal{D}}"', "\sim"] & \\
\mathrm{Fix}(RL) \arrow[d, equal] & & \mathrm{Fix}(LR) \arrow[d, equal] \\
\mathrm{Im}(R) \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\cong"] \arrow[d, hook] & & \mathrm{Im}(L) \arrow[ll, bend left, "R"] \arrow[d, hook] \\
\mathcal{C} \arrow[rr, bend left, "L"] & & \mathcal{D} \arrow[ll, bend left, "R"]
\end{tikzcd}$$

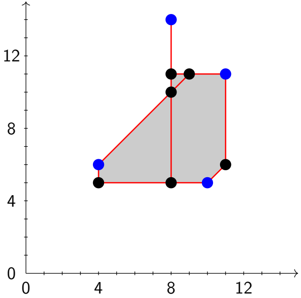
Tropical convex hull

The usual convex hull of points $\{p_i\}_{i=1}^s \subset \mathbb{R}^r$ is

$$\left\{ \sum_{i=1}^s \alpha_i p_i \mid \sum_{i=1}^s \alpha_i = 1,\ \alpha_i \in \mathbb{R}_+ \right\}.$$

In the tropical version $\oplus = \min$, $\cdot = +$ and the tropical convex hull is

$$\left\{ \bigoplus_{i=1}^s \alpha_i \cdot p_i \mid \min_{i=1}^s \alpha_i = 0,\ \alpha_i \in \mathbb{R}_+ \right\}.$$



The finite part of the set of tight price plans for receivers (where we need to look for optimal price plans) is the tropical convex hull set of costs to each supplier.

Isbell-type adjunction

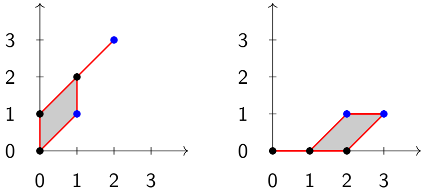
Given a $\mathcal{V}$-profunctor $f\colon \mathcal{C} \nrightarrow \mathcal{D}$, from the closed structure we get another adjunction

$$\begin{tikzcd} \mathcal{V}^{(\mathcal{C}^{\mathrm{op}})} \arrow[r, bend left, "-\triangleright k"] \arrow[r, phantom, "\perp"] & (\mathcal{V}^{\mathcal{D}})^{\mathrm{op}} \arrow[l, bend left, "-\triangleright k"] \end{tikzcd}$$

The centre of this adjunction (the profunctor nucleus) arises in other optimization and related areas.

- tight spans of metric spaces (server placement on networks)
- fuzzy concept analysis
- Legendre-Fenchel transform
- multi-commodity flow

$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ over $\overline{\mathbb{R}}_+$:



Star autonomy implies Kan-centres are Isbell-centres

Suppose $\mathcal{V}$ is closed monoidal and for some $d \in \mathcal{V}$ the map

$$v \mapsto [[v, d], d]$$

is an isomorphism for all v then $\mathcal{V}$ is star-autonomous.

$$(-)^{\star} = [-, d] \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\mathrm{op}}$$

- $\overline{\mathbb{R}}$ and Truth are star-autonomous,
- $\overline{\mathbb{R}}_{+}$ is not.

Given $k \colon \mathcal{C} \nrightarrow \mathcal{D}$ get $k^{\star} \colon \mathcal{C}^{\mathrm{op}} \nrightarrow \mathcal{D}^{\mathrm{op}}$.

If $\mathcal{V}$ is $\star$-autonomous, then all Kan-type centres arise as Isbell-type centres.

$$\begin{tikzcd}
\mathcal{V}^{\mathcal{C}} \arrow[r, bend left, "k \circ -"] \arrow[r, phantom, "\perp"] \arrow[d, "="] & \mathcal{V}^{\mathcal{D}} \arrow[l, bend left, "k \triangleright -"] \arrow[d, "(-)^{\star}"] \\
\mathcal{V}^{\mathcal{C}} \arrow[r, bend left, "- \triangleright k^{\star}"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathcal{D}^{\mathrm{op}}} \arrow[l, bend left, "- \triangleright k^{\star}"]
\end{tikzcd}$$

Summary

It seems that (enriched) category theory could be an organising structure for optimization and related areas similarly to how it is for other disciplines.