Magnitude of Metric Spaces II

Tom Leinster & Simon Willerton
Universities of Glasgow & Sheffield

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Weighting and magnitude

Recall:

- Suppose $A$ is a finite metric space.

A weighting is a function $w: A \to \mathbb{R}$ such that

$$\sum_{b \in A} e^{-d(a, b)} w_b = 1 \text{ for all } a \in A.$$ 

If a weighting exists then the magnitude is given by

$$|A| := \sum_{a \in A} w_a.$$ 

Think: Each $a \in A$ is an organism;
- wishes to be at temperature 1;
- generates $w_a$ amount of heat;
- experiences heat from $b$ as $e^{-d(a, b)} w_b$. 

$1/12$
Weighting and magnitude

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Recall: Infinite spaces and intervals

If $A$ is an infinite metric space define

$$|A| := \sup\left\{ |\tilde{A}| : \tilde{A} \subset A \text{ finite} \right\}$$
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For example

$$|\ell| = \ell/2 + 1$$
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$$|\ell/2 + 1| = \ell/2 + 1$$

Theorem (Leinster et al.): If $\bar{A} \subset \mathbb{R}^m$ is finite then $|\bar{A}|$ exists.
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Theorem (Leinster et al.): If $\bar{A} \subset \mathbb{R}^m$ is finite then $|\bar{A}|$ exists.

Theorem (Meckes): Suppose $A \subset \mathbb{R}^m$.
If $\{\bar{A}_i\}$ is a sequence of finite subsets of $A$ with $\bar{A}_i \to A$ then $|A_i| \to |A|$.
Homogeneous spaces and circles

**Lemma (Speyer):** Suppose $A$ is a homogeneous metric space. There is a constant weighting $w$: for any fixed $a_0 \in A$

$$w := \frac{1}{\sum_{a \in A} e^{-d(a_0, a)}}$$

so

$$|A| = \frac{\#A}{\sum_{a \in A} e^{-d(a_0, a)}}$$

For example

$$|C_n^\ell| \to \ell/2 \int_0^1 e^{-\ell s} ds \left[ n \to \infty \right]$$

So

$$|S_1^\ell| = \frac{\ell/2}{\int_0^1 e^{-\ell s} ds} \sim \frac{\ell/2}{\ell - 2} \left[ \ell \to \infty \right]$$
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**Lemma (Speyer):** Suppose $A$ is a **homogeneous** metric space. There is a constant weighting $w$: for any fixed $a_0 \in A$

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\[
|C^n_\ell| = \frac{n}{\sum_{a \in C^n_\ell} e^{-d(a_0, a)}}
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$C^n_\ell := \ldots$
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Approximating a square

We don’t know how to calculate the magnitude of subsets of $\mathbb{R}^2$. 
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We don’t know how to calculate the magnitude of subsets of $\mathbb{R}^2$. Approximate with a finite subset and get maple to calculate a weighting.
Bulk approximation heuristic

Let $\mathcal{L}$ be a ‘small’ lattice in $\mathbb{R}^m$. 

$$w = \sum_{a \in \mathcal{L}} \exp\left(-d(0,a)\right) \approx \int_{x \in \mathbb{R}^m} \exp(-|x|) \, \text{dvol} = \omega_m$$

Suppose $A \subset \mathbb{R}^m$ is ‘large’ and the closure of an open subset. Contribution to $|A \cap \mathcal{L}|$ due to the ‘bulk’ far from the boundary is ‘roughly’

$$\sum_{a \in \text{bulk}} \omega_m \sim m^{5/12}$$
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Let $\mathcal{L}$ be a ‘small’ lattice in $\mathbb{R}^m$. Homogeneous so has a weighting.
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$$\sum_{a \in \text{bulk}} \frac{\text{vol } \Delta}{m! \omega_m} \sim \frac{\mu_m A}{m! \omega_m}$$
The valuation $P$

Define the valuation $P$ of compact subset $A \subset \mathbb{R}^m$

$$P(A) := \sum_{i=0}^{m} \frac{\mu_i(A)}{i! \omega_i}$$
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$$P(A) := \sum_{i=0}^{m} \frac{\mu_i(A)}{i! \omega_i} = \frac{\mu_m A}{m! \omega_m} + \cdots + \frac{\mu_2 A}{2\pi} + \frac{\mu_1 A}{2} + \chi A.$$
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Let $\bar{A} \subset A$ mean a finite subset.

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Let \( \check{A} \subset A \) mean a finite subset.

Guess.

- For \( \check{A} \) a reasonable approximation: \( |\check{A}| \simeq |A| \).
Define the valuation $P$ of compact subset $A \subset \mathbb{R}^m$

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Let $\tilde{A} \subset A$ mean a finite subset.

**Guess.**

- For $\tilde{A}$ a reasonable approximation: $|\tilde{A}| \simeq |A|$.
- For $A$ large and closure of an open set: $|\tilde{A}| \simeq P(A)$ [bulk approximation].
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**Test the guess.**
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- Pick some simple subset $A$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a scale factor $t > 0$. 

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- Pick some simple subset $A$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a scale factor $t > 0$.  
- Calculate $P(tA)$.  


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Let \( \tilde{A} \subset A \) mean a finite subset.

Guess.

- For \( \tilde{A} \) a reasonable approximation: \( |\tilde{A}| \simeq |A| \).
- For \( A \) large and closure of an open set: \( |\tilde{A}| \simeq P(A) \) [bulk approximation].

Test the guess.

- Pick some simple subset \( A \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) and a scale factor \( t > 0 \).
- Calculate \( P(tA) \).
- Get a computer to calculate \( |t\tilde{A}| \) for an approximation \( \tilde{A} \).
The valuation $P$

Define the **valuation $P$** of compact subset $A \subset \mathbb{R}^m$

$$P(A) := \sum_{i=0}^{m} \frac{\mu_i(A)}{i! \omega_i} = \frac{\mu_mA}{m! \omega_m} + \cdots + \frac{\mu_2A}{2\pi} + \frac{\mu_1A}{2} + \chi A.$$  

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- Pick some simple subset $A$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a scale factor $t > 0$.
- Calculate $P(tA)$.
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- Compare the two!
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- Calculate $P(tA)$.
- Get a computer to calculate $|t\bar{A}|$ for an approximation $\bar{A}$.
- Compare the two!
- Repeat.
Some calculations

Squares:

Discs:

Cubes:

Annuli:
Some calculations

Squares:

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Fractals: Ternary Cantor sets

\[ T_\ell^0 := \cdot \ell \cdot \]

The length of the ternary Cantor set is the limit of these sets:

\[ T_k \ell \rightarrow T_\ell \]

It is easy to calculate the magnitudes of the approximations:

\[ |T_k \ell| \quad \text{where} \quad f(3 \ell) = f(\ell) \quad \text{and} \quad f(\ell) \approx 1.205. \]
Fractals: Ternary Cantor sets

\[ T^1_\ell := \quad \cdots \quad . \quad \cdots \quad \ell \quad \cdots \quad . \]
Fractals: Ternary Cantor sets

\[ T_\ell^2 := \ldots \ldots \ldots \ldots \ldots \]

The length of the ternary Cantor set is the limit of these sets:

\[ T_k^\ell \rightarrow T_\ell^\ell \]

It is easy to calculate the magnitudes of the approximations:

\[ |T_k^\ell| \] (where \( f(3\ell) = f(\ell) \) and \( f(\ell) \approx 1.205 \).)
Fractals: Ternary Cantor sets

\[ T_\ell^3 := \ldots \quad \ldots \quad \ldots \quad \ldots \]
Fractals: Ternary Cantor sets

\[ T_\ell^3 := \ldots \ldots \ldots \ldots \ldots \]

The length \( \ell \) ternary Cantor set is the limit of these sets:
The length $\ell$ ternary Cantor set is the limit of these sets: $T^3_\ell \to T_\ell$
Fractals: Ternary Cantor sets

\[ T^3_\ell := \begin{array}{c\ldots c\ldots c\ldots c\ldots c} \ell \end{array} \]

The length \( \ell \) ternary Cantor set is the limit of these sets: \( T^k_\ell \rightarrow T_\ell \)

It is easy to calculate the magnitudes of the approximations:

\[ |T^k_\ell| = 1 + \frac{1}{2} \sum_{i=1}^{k} 2^i \tanh \left( \frac{\ell}{2 \cdot 3^i} \right) + 2^k \tanh \left( \frac{\ell}{2 \cdot 3^k} \right) \]
Fractals: Ternary Cantor sets

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It is easy to calculate the magnitudes of the approximations:

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Fractals: Ternary Cantor sets

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Fractals: Ternary Cantor sets

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It is easy to calculate the magnitudes of the approximations:

\[ |T_\ell| = f(\ell) \cdot \ell^{\log_3 2} + O(\ell^{-1}) \quad \text{as } \ell \to \infty \]

(\( f(3\ell) = f(\ell) \) and \( f(\ell) \approx 1.205 \).)
Fractals: Ternary Cantor sets

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(where $f(3\ell) = f(\ell)$ and $f(\ell) \simeq 1.205$.)

**Lemma:** Suppose $p$ is a function on $\{T_\ell\}$ then $p$ satisfies the inclusion-exclusion principle if and only if

$$p(T_\ell) = f(\ell) \cdot \ell^\log_3 2$$

for some $f: (0, \infty) \to \mathbb{R}$ with $f(3\ell) = f(\ell)$. 
Fractals: Ternary Cantor sets

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(where \( f(3\ell) = f(\ell) \) and \( f(\ell) \approx 1.205 \).)
Convex Conjecture: If $K \in \mathbb{R}^m$ is a convex set then $|K| = \mathcal{P}(K)$.

Asymptotic Principle: There is a large class $C$ of compact subsets of Euclidean space and a function $p: C \to \mathbb{R}$ which is tractable and interesting, possibly related to valuations, such that for $A \in C$, $|tA| \simeq p(tA)$ as $t \to \infty$.

For example:
- finite sets of points: $p = \text{cardinality} = \mathcal{P}$
- circles: $p = \frac{1}{2} \text{half the circumference} = \mathcal{P}$
- finite unions of intervals in the line: $p = \mathcal{P}$
- Cantor sets: $p(T_\ell) = f(\ell) \cdot \ell \log_2 3$

Guess: For $A$ the closure of an open set $p(A) = \mathcal{P}(A)$. 

Euclidean subspaces: summary
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Convex Conjecture: If $K \in \mathbb{R}^m$ is a convex set then

$$|K| = P(K).$$
Euclidean subspaces: summary

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For example

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Guess: For $A$ the closure of an open set $p(A) = P(A)$. 
Measure theoretic approach

A weight measure for \( A \) is a signed measure \( \nu \) such that:

\[
\int_{b \in A} e^{-d(a, b)} \, d\nu_b = 1 \quad \text{for all} \quad a \in A.
\]

If a weight measure \( \nu \) exists then the measure magnitude is defined by:

\[
\|A\| := \int_A d\nu.
\]

Eg: For \( L_{\ell} := \ell \quad \frac{1}{2}(\mu + \delta_0 + \delta_{\ell}) \).

Hence:

\[
\|L_{\ell}\| = \int_{L_{\ell}} \frac{1}{2}(d\mu + d\delta_0 + d\delta_{\ell}).
\]

Theorem (Meckes): If \( A \subset \mathbb{R}^m \) and \( \|A\| \) exists then:

\[
\|A\| = |A|.
\]
Measure theoretic approach

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If a weight measure $\nu$ exists then the measure magnitude is defined by

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\textbf{Eg:}  For $L_\ell := \ell \mu + \delta_0 + \delta_\ell$ a weight measure is $\frac{1}{2}(\mu + \delta_0 + \delta_\ell)$.  

Hence

$$\|L_\ell\| = \frac{1}{2}(\ell + 1 + 1)$$
Measure theoretic approach

A **weight measure** for \( A \) is a signed measure \( \nu \) such that that

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\]

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\[
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\]

**Eg:** For \( L_\ell := \frac{1}{2}(\mu + \delta_0 + \delta_\ell) \) a weight measure is \( \frac{1}{2}(\mu + \delta_0 + \delta_\ell) \).

Hence

\[
\| L_\ell \| = \ell/2 + 1.
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Measure theoretic approach

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Hence

$$\|L_\ell\| = \ell/2 + 1.$$

Theorem (Meckes): If $A \subset \mathbb{R}^m$ and $\|A\|$ exists then $\|A\| = |A|$. 
Homogeneous manifolds: measure magnitude

Suppose $A$ a homogeneous metric space and $\mu$ an invariant measure. There is weight measure $\nu$ on $A$: for any fixed $a \in A$

$$\nu := \frac{\mu}{\int_{b \in A} e^{-d(a,b)} d\mu_b}$$
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$$\nu := \frac{\mu}{\int_{b \in A} e^{-d(a,b)} d\mu_b} \quad \text{so} \quad \|A\| = \frac{\int_{A} d\mu}{\int_{b \in A} e^{-d(a,b)} d\mu_b}.$$
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Suppose $X$ is a homogeneous Riemannian manifold.
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Suppose $X$ is a homogeneous Riemannian manifold.

- It has the geodesic metric.
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Suppose $X$ is a homogeneous Riemannian manifold.

- It has the geodesic metric.
- It has an invariant measure from the volume form.
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$$

Suppose $X$ is a homogeneous Riemannian manifold.

- It has the geodesic metric.
- It has an invariant measure from the volume form.

So

$$
\|X\| = \frac{\text{vol}(X)}{\int_X e^{-d(a, b)} d\text{vol}_b}.
$$
Homogeneous manifolds: spheres

Suppose $S^n_R$ is the radius $R$ sphere with the geodesic metric.
Homogeneous manifolds: spheres

Suppose $S^n_R$ is the radius $R$ sphere with the geodesic metric.

$$\|S^n_R\| = \begin{cases} 
\frac{2\left((\frac{R}{n-1})^2 + 1\right)\left((\frac{R}{n-3})^2 + 1\right) \ldots \left((\frac{R}{1})^2 + 1\right)}{1 + e^{-\pi R}} & \text{for } n \text{ even} \\
\frac{\pi R\left((\frac{R}{n-1})^2 + 1\right)\left((\frac{R}{n-3})^2 + 1\right) \ldots \left((\frac{R}{2})^2 + 1\right)}{1 - e^{-\pi R}} & \text{for } n \text{ odd}
\end{cases}$$
Homogeneous manifolds: spheres

Suppose $S^n_R$ is the radius $R$ sphere with the geodesic metric.

\[
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\end{cases}
\]

Theorem (Meckes): \(\|S^n_R\| = |S^n_R|\).
Homogeneous manifolds: spheres

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$$
\|S^n_R\| = \begin{cases} 
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\frac{1}{1 + e^{-\pi R}} 
\end{cases}
$$

for $n$ even

$$
\pi R \left( \left( \frac{R}{n-1} \right)^2 + 1 \right) \left( \left( \frac{R}{n-3} \right)^2 + 1 \right) \ldots \left( \left( \frac{R}{2} \right)^2 + 1 \right) \\
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$$

for $n$ odd
Homogeneous manifolds: spheres

Suppose $S^n_R$ is the radius $R$ sphere with the geodesic metric.

\[ \|S^n_R\| = \frac{\mu_n(S^n_R)}{n! \omega_n} + 0 + \left[ \frac{(n+1)}{3(n-1)} \right] \frac{\mu_{n-2}(S^n_R)}{(n-2)! \omega_{n-2}} + 0 + \cdots + \chi(S^n_R) + O(R^{-1}) \quad \text{as } R \to \infty. \]
Homogeneous manifolds: asymptotics

Suppose $X^n$ is a homogeneous Riemannian manifold, $t > 0$.

$$\|tX\| = \frac{\text{vol}(tX)}{\int_X e^{-td(a,b)} \text{dvol}_b}$$
Homogeneous manifolds: asymptotics

Suppose $X^n$ is a homogeneous Riemannian manifold, $t > 0$.

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Key points:

▶ The scalar curvature $\tau(x)$ measures the lack of 'stuff' near $x$.

▶ $\mu_{n-2}(X) = \frac{1}{4\pi} \int_X \tau(x) \, d\text{vol}$

For example, suppose $\Sigma$ is a homogeneous Riemannian 2-sphere or 2-torus:

$$\|t\Sigma\| = \frac{\text{Area}(t\Sigma)}{2\pi} + \chi(t\Sigma) + O(t^{-2}) \text{ as } t \to \infty.$$
Homogeneous manifolds: asymptotics

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For example, suppose $\Sigma$ is a homogeneous Riemannian 2-sphere or 2-torus, then

$$\|t\Sigma\| = \frac{\text{Area}(t\Sigma)}{2\pi} + \frac{\chi(t\Sigma)}{4\pi} + O(t^{-2})$$

as $t \to \infty$. 

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Homogeneous manifolds: asymptotics

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Suppose $X^n$ is a homogeneous Riemannian manifold, $t > 0$.

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Suppose $\Sigma$ is a homogeneous Riemannian 2-sphere or 2-torus

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\|t\Sigma\| = \frac{\text{Area}(t\Sigma)}{2\pi} + \chi(t\Sigma) + O(t^{-2}) \quad \text{as } t \to \infty.
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